Equilibrium existence results for a class of discontinuous games

Monica Patriche

University of Bucharest Faculty of Mathematics and Computer Science
14 Academiei Street
010014 Bucharest, Romania
monica.patriche@yahoo.com

Abstract

We introduce the notions of w-lower semicontinuous and almost w-lower semicontinuous correspondence with respect to a given set and prove a new fixed-point theorem. We also introduce the notion of correspondence with e-LSCS-property. As applications we obtain some new equilibrium theorems for abstract economies and for generalized multiobjective games.

Keywords: w-lower semicontinuous correspondences, correspondences with e-LSCS-property, abstract economy, equilibrium, generalized weighted Nash equilibrium, generalized Pareto equilibrium, generalized multiobjective game.

1. Introduction

In [16] W. Shafer and H. Sonnenschein proved the existence of equilibrium of an economy with finite dimensional commodity space and irreflexive preferences represented as correspondences with open graph. They generalized the work of J. Nash [14], who first proved a theorem of equilibrium existence for games where the player's preferences are representable by continuous quasi-concave utilities and the work of G. Debreu, who proved the existence of equilibrium in a generalized N-person game or an abstract economy [3]. N. C. Yannelis and N. D. Prahbakar [20] developed new techniques based on selection theorems and fixed-point theorems. Their main result concerns the existence of equilibrium when the constraint and preference correspondences

have open lower sections. They worked within different framework (countable infinite number of agents, infinite dimensional strategy spaces). K. J. Arrow and G. Debreu proved the existence of Walrasian equilibrium in [1]. To sum up, the significance of equilibrium theory stems from the fact that it develops important tools to prove the existence of equilibrium for different types of games.

A. Borglin and H. Keiding [2] used new concepts of K. F.-correspondences and K. F.-majorized correspondences for their existence results. The second notion was extended by Yannelis and Prabhakar [20] to L-majorized correspondences. In [21], G. X. Yuan proposed a model of abstract economy more general than that introduced by A. Borglin and H. Keiding in [2], meaning that each constraint mapping has been divided into two parts, A and B, because the set of the fixed points of the "small" correspondence may not be rich enough.

Most of the existence theorems of equilibrium deal with preference correspondences which have lower open sections or are majorized by correspondences with lower open sections. Within the last years, some existence results were obtained for lower semicontinuous and upper semicontinuous correspondences. Some results concerning fixed point theorems for lower semicontinuous correspondences or equilibrium existence for economies with lower semicontinuous and Q-majorized correspondences can be found in [4], [5], [10], [18]. E. Michael gave some selection theorems for lower semicontinuous correspondences. His main results can be found in [11]-[13].

In this paper, we define several types of correspondences: w-lower semicontinuous and almost w-lower semicontinuous with respect to a given set and also correspondences having e-LSCS-property. We prove a fixed point theorem for almost w-lower semicontinuous correspondences. This result is a Wu like fixed point theorem [18]. We use this theorem to prove the equilibrium existence for abstract economies which have w-lower semicontinuous constraint correspondences. We use a technique of approximation to prove an equilibrium existence theorem for correspondences with e-LSCS-property.

We give slight generalizations of the equilibrium notions defined by W. K. Kim and X. P. Ding in [9] and we also prove the existence of generalized weighted Nash equilibrium and of generalized Pareto equilibrium for a generalized multiobjective game having w-lower semicontinuous constraints.

The paper is organized in the following way: Section 2 contains preliminaries and notation. The fixed point theorem is presented in Section 3 and the equilibrium theorems are stated in Section 4. Section 5 contains

the model of a constrained multiobjective game and a Pareto equilibrium existence result.

2. Preliminaries and notation

We shall denote by $\mathbb{R}_+^m := \{u = (u_1, u_2, ..., u_m) \in \mathbb{R}^m : u_j \geq 0 \ \forall j = 1, 2, ..., m\}$ and $\text{int}\mathbb{R}_+^m := \{u = (u_1, u_2, ..., u_m) \in \mathbb{R}^m : u_j > 0 \ \forall j = 1, 2, ..., m\}$ the non-negative othant of $\mathbb{R}^m$ and respective the non-empty interior of $\mathbb{R}_+^m$ with the topology induced in terms of convergence of vector with respect to the Euclidian metric. For each $u, v \in \mathbb{R}^m$, $u \cdot v$ denote the standard Euclidian inner product.

Now we present some notations and results concerning the theory of correspondences.

Let A be a subset of a topological space X. $F(A)$ denotes the family of all nonempty finite subset of A. 2^A denotes the family of all subsets of A. cl$A$ denotes the closure of A in X. If A is a subset of a vector space, co$A$ denotes the convex hull of A. If F, $G : X \to 2^Y$ are correspondences, then co$G$, cl$G$, $G \cap F : X \to 2^Y$ are correspondences defined by $(\text{co}G)(x) = \text{co}G(x)$, $(\text{cl}G)(x) = \text{cl}G(x)$ and $(G \cap F)(x) = G(x) \cap F(x)$ for each $x \in X$, respectively. The graph of $T : X \to 2^Y$ is the set $\text{Gr}(T) = \{(x, y) \in X \times Y \mid y \in T(x)\}$.

The correspondence $\overline{T}$ is defined by $\overline{T}(x) = \{y \in Y : (x, y) \in \text{cl}_{X \times Y}\text{Gr}T\}$ (the set $\text{cl}_{X \times Y}\text{Gr}(T)$ is called the adherence of the graph of T). It is easy to see that $\text{cl}T(x) \subset \overline{T}(x)$ for each $x \in X$.

Remark 1. *$\overline{T}(x) = clT(x)$ for each $x \in X$ if T has a closed graph in $X \times Y$ (by Theorem 7.1.15 in [8], it follows that in particular, T has a closed graph when Y is regular and clT is upper semicontinuous with closed values).*

Let X, Y be topological spaces and $T : X \to 2^Y$ be a correspondence. T is said to be *lower semicontinuous (l.s.c)* if for each x$\in X$ and each open set V in Y with $T(x) \cap V \neq \emptyset$, there exists an open neighbourhood U of x in X such that $T(y) \cap V \neq \emptyset$ for each $y \in U$. $T : X \to 2^Y$ is said to be *almost lower semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \cap V \neq \phi$, there exists an open neighborhood U of x in X such that $T(x) \cap \overline{V} \neq \phi$ for each $z \in U$.

For some known results about lower semicontinuity, which will be used in our proofs, we refer the reader to [8].

Proposition 1. *(Lemma 1 in [17], Proposition 2.5 in [11]). Let X and Y be topological spaces and let T_1, T_2 be two l.s.c correspondences from X to Y. If T_1 has open values and $T_1(x) \cap T_2(x) \neq \emptyset$, for every $x \in X$, then, the correspondence T defined by $T(x) = T_1(x) \cap T_2(x)$ is l.s.c too.*

Proposition 2. *(Theorem 1.6, pag 25 in [21]). Let X be a topological space, E be a topological vector space and Y be a non-empty subset of E. Suppose $S : X \to 2^Y$ is a lower semicontinuous correspondence and V is any nonempty open subset of E. Then the correspondence $T : X \to 2^Y$ defined by $T(x) = (S(x) + V) \cap Y$ for each $x \in X$ has an open graph in $X \times Y$.*

We also need a version of Lemma 1.1 in [21]. For the reader's convenience, we include its proof below.

Lemma 3. *Let X be a topological space, Y be a nonempty subset of a locally convex topological vector space E and $T : X \to 2^Y$ be a correspondence. Let ß be a basis of neighbourhoods of 0 in E consisting of open absolutely convex symmetric sets. Let D be a compact subset of Y. If for each $V \in$ß, the correspondence $T^V : X \to 2^Y$ is defined by $T^V(x) = (T(x) + V) \cap D$ for each $x \in X$, then $\cap_{V \in ß}\overline{T^V}(x) \subseteq \overline{T}(x)$ for every $x \in X$.*

Proof. Let be x and y be such that $y \in \cap_{V \in ß}\overline{T^V}(x)$ and suppose, by way of contradiction, that $y \notin \overline{T}(x)$. This means that $(x, y) \notin$clGrT, so that there exists an open neighborhood U of x and $V \in$ß such that:

$(U \times (y + V)) \cap \mathrm{Gr}T = \emptyset$. (1)

Choose $W \in$ß such that $W - W \subseteq V$ (e.g. $W = \frac{1}{2}V$). Since $y \in \overline{T^W}(x)$, then $(x, y) \in$clGrT^W, so that

$$(U \times (y + W)) \cap \mathrm{Gr}T^W \neq \emptyset.$$

There are some $x' \in U$ and $w' \in W$ such that $(x', y + w') \in$GrT^W, i.e. $y + w' \in T^W(x')$. Then, $y + w' \in D$ and $y + w' = y' + w''$ for some $y' \in T(x')$ and $w'' \in W$. Hence, $y' = y + (w' - w'') \in y + (W - W) \subseteq y + V$, so that $T(x') \cap (y + V) \neq \emptyset$. Since $x' \in U$, this means that $(U \times (y + V)) \cap \mathrm{Gr}T \neq \emptyset$, contradicting (1).

We present first Wu's Theorem 1 in [18], which will be generalized in the next section.

Theorem 4 (18). *Let I be an index set. For each $i \in I$, let X_i be a nonempty convex subset of a Hausdorff locally convex topological vector space E_i, D_i a non-empty compact metrizable subset of X_i and $S_i, T_i : X \to 2^{D_i}$ two correspondences with the following conditions:*

1) *for each $x \in X := \prod_{i \in I} X_i$, $\mathrm{clco} S_i(x) \subset T_i(x)$ and $S_i(x) \neq \emptyset$;*
2) *S_i is lower semicontinuous.*
Then there exists $x^* \in D = \prod_{i \in I} D_i$ such that $x_i^* \in T_i(x^*)$ for each $i \in I$.

In the present paper, our purpose is to give a fixed point theorem and to research the equilibrium existence problem for abstract economies. In order to establish our main results, we introduce the following definitions.

Let X be a topological space, Y be a nonempty subset of a topological vector space E and D be a subset of Y.

Definition 1. *The correspondence $T : X \to 2^Y$ is said to be w-lower semicontinuous (weakly lower semicontinuous) with respect to D if there exists a basis ß of open symmetric neighbourhoods of 0 in E such that, for each $V \in$ß, the correspondence T^V is lower semicontinuous, where $T^V(x) = (T(x) + V) \cap D$ for each $x \in X$,.*

Remark 2. *By Lemma 2.6 in [19], it follows that if the correspondence $T : X \to 2^Y$ is almost lower semicontinuous, then it is w-lower semicontinuous with respect to Y.*

Definition 2. *The correspondence $T : X \to 2^Y$ is said to be almost w-lower semicontinuous (almost weakly lower semicontinuous) with respect to D if there exists a basis ß of open symmetric neighbourhoods of 0 in E such that, for each $V \in$ß, the correspondence $\overline{T^V}$ is lower semicontinuous.*

Example 1. *Let $T_1 : (0, 2) \to 2^{[1,4]}$ be the correspondence defined by*

$$T_1(x) = \begin{cases} [2-x, 2], & \text{if } x \in (0,1); \\ \{4\} & \text{if } \quad x = 1; \\ [1,2] & \text{if } \quad x \in (1,2). \end{cases}$$

T_1 is not lower semicontinuous on $(0, 2)$.

Let $D = [1, 2]$ and let $V = (-\varepsilon, \varepsilon)$, $\varepsilon > 0$, be an open symmetric neighbourhood of 0 in $\mathbb{R}$. Then, it results that

for $\varepsilon \in (0,1)$,

$$T_1^V(x) = (T_1(x) + (-\varepsilon,\varepsilon)) \cap D = \begin{cases} (2-x-\varepsilon, 2], & \text{if } x \in (0, 1-\varepsilon]; \\ [1,2] & \text{if } x \in (1-\varepsilon, 1) \cup (1,2); \\ \phi & \text{if } x = 1; \end{cases},$$

for $\varepsilon \in [1,2]$,

$$T_1^V(x) = (T_1(x) + (-\varepsilon,\varepsilon)) \cap D = \begin{cases} [1,2], & \text{if } x \in (0,1) \cup (1,2); \\ \phi & \text{if } x = 1; \end{cases}$$

if $\varepsilon \in (2,3]$,

$$T_1^V(x) = (T_1(x) + (-\varepsilon,\varepsilon)) \cap D = \begin{cases} [1,2] & \text{if } x \in (0,1) \cup (1,2); \\ (4-\varepsilon, 2] & \text{if } x = 1; \end{cases}$$

and if $\varepsilon > 3$,

$T_1^V(x) = (T_1(x) + (-\varepsilon,\varepsilon)) \cap D = [1,2]$ if $x \in (0,2)$.

Then,

for $\varepsilon \in (0,1)$,

$$\overline{T_1^V}(x) = \begin{cases} [2-x-\varepsilon, 2], & \text{if } x \in [0, 1-\varepsilon); \\ [1,2] & \text{if } x \in (1-\varepsilon, 2]; \end{cases} \quad \text{and}$$

for $\varepsilon \geq 1$,

$\overline{T_1^V}(x) = [1,2]$ for $x \in [0,2]$.

For each $V = (-\varepsilon,\varepsilon)$ with $\varepsilon > 0$, the correspondences T_1^V and $\overline{T_1^V}$ are lower semicontinuous and $\overline{T_1^V}$ has nonempty values. We conclude that T_1 is w-lower semicontinuous with respect to D and it is also almost w-lower semicontinuous with respect to D.

Proposition 5. *Let X be a topological space, Y be a nonempty subset of a topological vector space E. If the correspondence $T : X \to 2^Y$ is lower semicontinuous and nonempty valued, then it is also w-lower semicontinuous with respect to any set $D \subset Y$ with the property that $T(x) \cap D \neq \phi$, for every $x \in X$.*

Proof. Let V be an open symmetric neighborhood of 0 in E. Since the constant valued correspondence $x \to V$ is lower semicontinuous, it follows that so it is the correspondence $x \to (T(x) + V)$. Note also that this correspondence has nonempty open values and that $(T(x) + V) \cap D \neq \emptyset$ for every $x \in X$. Further, Proposition 1 can be applied for $T_1(x) = T(x) + V$ and $T_2(x) = D$, $x \in X$.

Remark 3. *If the corespondence T^V has empty values for some open set V, it may not be lower semicontinuous. The following example proves this assertion.*

Example 2. *Let* $T_1 : [0, 2] \to 2^{[3,5)}$ *be the correspondence defined by*

$$T_1(x) = \begin{cases} [x + 2, 4] & \text{if } x \in [0, 1]; \\ (4, 5) & \text{if } x \in (1, 2]; \end{cases}.$$

Let $V = (-1, 1)$ and $D = [0, 3]$. Then,

$$T_1^{(-1,1)}(x) = \begin{cases} (x + 1, 5) & \text{if } x \in [0, 1]; \\ (3, 6) & \text{if } x \in (1, 2]; \end{cases} \cap [0, 3] = \begin{cases} (x + 1, 3] & \text{if } x \in [0, 1]; \\ \phi & \text{if } x \in (1, 2]; \end{cases}$$

$T_1^{(-1,1)}$ is not lower semicontinuous.

Remark 4. $\overline{T^V}$ *may not have convex values, even if* T^V *is convex valued.*

Example 3. *Let* $D = [1, 2]$ *and* $T : [0, 2] \to 2^{[0,4)}$ *be the correspondence defined by*

$$T(x) = \begin{cases} [0, 1] & \text{if } x \in [0, 1); \\ \phi & \text{if } x = 1; \\ (2, 3) & \text{if } x \in (1, 2]. \end{cases}$$

T is lower semicontinuous on $[0, 2]$.

If $\varepsilon \in (0, \frac{1}{2})$, $T^V(x) = \begin{cases} [1, 1 + \varepsilon) & \text{if } x \in [0, 1); \\ \phi & \text{if } x = 1; \\ (2 - \varepsilon, 2] & \text{if } x \in (1, 2]. \end{cases}$

Then, if $\varepsilon \in (0, \frac{1}{2})$, $\overline{T^V}(x) = \begin{cases} [1, 1 + \varepsilon] & \text{if } x \in [0, 1); \\ [1, 1 + \varepsilon] \cup [2 - \varepsilon, 2] & \text{if } x = 1; \\ [2 - \varepsilon, 2] & \text{if } x \in (1, 2]; \end{cases}$ for $V = (-\varepsilon, \varepsilon)$.

$\overline{T^V}$ does not have convex values in every point $x \in [0, 2]$.

We also define the dual w-lower semicontinuity with respect to a set.

Definition 3. *Let* $T_1, T_2 : X \to 2^Y$ *be correspondences. The pair* (T_1, T_2) *is said to be dual almost w-lower semicontinuous (dual weakly lower semicontinuous) with respect to* D *if there exists a basis ß of open symmetric neighbourhoods of* 0 *in* E *such that, for each* $V \in$ß*, the correspondence* $\overline{T^V_{(1,2)}} : X \to 2^D$ *is lower semicontinuous, where* $T^V_{(1,2)} : X \to 2^D$ *is defined by* $T^V_{(1,2)}(x) = (T_1(x) + V) \cap T_2(x) \cap D$ *for each* $x \in X$.

Example 4. *Let* $D = [1, 2]$, $T_1 : (0, 2) \to 2^{[1,4]}$ *be the correspondence from the example 1 and* $T_2 : (0, 2) \to 2^{[2,3]}$ *be the correspondence defined by*

$T_2(x) = \begin{cases} [2,3], \text{ if } x \in (0,1]; \\ \{2\} \text{ if } \ x \in (1,2); \end{cases}$.

The correspondences T_1 and T_2 are not semicontinuous.

For $\varepsilon \in (0,2]$, $(T_1(x)+(-\varepsilon,\varepsilon)) \cap D \cap T_2(x) = \begin{cases} \{2\} \text{ if } x \in (0,1) \cup (1,2); \\ \phi \quad \text{if} \quad x = 1. \end{cases}$

For $\varepsilon \in (2,\infty)$, $(T_1(x)+(-\varepsilon,\varepsilon)) \cap D \cap T_2(x) = \{2\}$ for each $x \in (0,2)$.

Then, we have that for each $\varepsilon > 0$, $\overline{T^V_{(1,2)}}(x) = \{2\}$ for each $x \in [0,2]$ and the correspondence $\overline{T^V_{(1,2)}}$ is lower semicontinuous and has nonempty values.

We conclude that the pair (T_1, T_2) is dual almost w-lower semicontinuous with respect to D.

3. A new fixed point theorem

We obtain a fixed point theorem which is an extension of Wu's fixed point Theorem 1 in [18], in the sense that, for each $i \in I$, the involved correspondence S_i is assumed to be almost w-lower semicontinuous with respect to a set D_i, but D_i is not convex as in the quoted result.

Theorem 6. *Let I be an index set. For each $i \in I$, let X_i be a nonempty convex subset of a Hausdorff locally convex topological vector space E_i, D_i be a nonempty compact convex metrizable subset of X_i and $S_i, T_i : X := \prod_{i\in I} X_i \to 2^{X_i}$ be two correspondences with the following conditions:*

1) *for each $x \in X$, $\overline{S}_i(x) \subset T_i(x)$.*

2) *S_i is almost w-lower semicontinuous with respect to D_i and $\overline{S_i^{V_i}}$ is convex nonempty valued for each open absolutely convex symmetric neighbourhood V_i of 0 in E_i.*

Then there exists $x^* \in D := \prod_{i\in I} D_i$ such that $x^*_i \in T_i(x^*)$ for each $i \in I$.

Proof. Since D_i is compact, $D := \prod_{i\in I} D_i$ is also compact in X. For each $i \in I$, let ß$_i$ be a basis of open absolutely convex symmetric neighbourhoods of zero in E_i and let ß$=\prod_{i\in I}$ß$_i$. For each system of neighbourhoods $V = (V_i)_{i\in I} \in \prod_{i\in I}$ß$_i$, let's define the corespondences $S_i^{V_i} : X \to 2^{D_i}$, by $S_i^{V_i}(x) = (S_i(x)+V_i) \cap D_i$, $x \in X$, $i \in I$. By assumption 2) each $\overline{S_i^{V_i}}$ is l.s.c with nonempty closed convex values. According to Theorem 1.1 in [12], there exists a nonempty valued, upper semicontinuous correspondence $G_i^{V_i} : D \to 2^{D_i}$

such that $G_i^{V_i}(x) \subset \overline{S_i^{V_i}}(x)$ for all $x \in D$. Then, by Theorem 7.3.5 in [8] and Theorem 1.4 pag. 25 in [21], the correspondence $F_i^{V_i}$ =clco$G_i^{V_i} : D \to 2^{D_i}$ is also upper semicontinuous with nonempty closed convex values. Let's define $F^V : D \to 2^D$ by $F^V(x) = \prod\limits_{i\in I} F_i^{V_i}(x)$ for each $x \in D$. The correspondence F^V is upper semicontinuous with closed convex values. Therefore, according to Himmelberg's fixed point theorem [7], there exists $x_V^* = \prod\limits_{i\in I} x_{V_i}^* \in D$ such that $x^* \in F^V(x^*)$. It follows that $x_{V_i}^* \in \overline{S_i^{V_i}}(x_V^*)$ for each $i \in I$.

For each $V = (V_i)_{i\in I} \in$ß, let's define $Q_V = \cap_{i\in I}\{x \in D : x_i \in \overline{S_i^{V_i}}(x)\}$.

Q_V is nonempty since $x_V^* \in Q_V$, then Q_V is nonempty and closed.

We prove that the family $\{Q_V : V \in$ ß$\}$ has the finite intersection property.

Let $\{V^{(1)}, V^{(2)}, ..., V^{(n)}\}$ be any finite set of ß and let $V^{(k)} = \prod\limits_{i\in I} V_i^{(k)}$, $k = 1, ..., n$. For each $i \in I$, let $V_i = \bigcap\limits_{k=1}^{n} V_i^{(k)}$, then $V_i \in$ ß$_i$; thus $V = \prod\limits_{i\in I} V_i \in \prod\limits_{i\in I}$ß$_i$. Clearly $Q_V \subset \bigcap\limits_{k=1}^{n} Q_{V^{(k)}}$ so that $\bigcap\limits_{k=1}^{n} Q_{V^{(k)}} \neq \emptyset$.

Since D is compact and the family $\{Q_V : V \in$ ß$\}$ has the finite intersection property, we have that $\cap\{Q_V : V \in$ ß$\} \neq \emptyset$. Take any $x^* \in \cap\{Q_V : V \in$ß$\}$, then for each $V_i \in$ ß$_i$, $x_i^* \in \overline{S_i^{V_i}}(x^*)$. Acording to Lemma 1, we have that $x_i^* \in \overline{S_i}(x^*)$, for each $i \in I$, therefore $x_i^* \in T(x^*)$.

If $|I| = 1$ we get the result bellow.

Corollary 7. *Let X be a nonempty convex subset of a Hausdorff locally convex topological vector space F, D be a nonempty compact convex metrizable subset of X and $S, T : X \to 2^X$ be two correspondences with the following conditions:*

1) *for each $x \in X$, $\overline{S}(x) \subset T(x)$ and $S(x) \neq \emptyset$,*

2) *S is almost w-lower semicontinuous with respect to D and $\overline{S^V}$ is convex nonempty valued for each open absolutely convex symmetric neighbourhood V of 0 in E.*

Then, there exists a point $x^* \in D$ such that $x^* \in T(x^*)$.

In the particular case that the correspondence $S = T$ the following result stands.

Corollary 8. *Let X be a nonempty convex subset of a Hausdorff locally convex topological vector space F, D be a nonempty compact convex metrizable subset of X and $T : X \to 2^X$ be an almost w- lower semicontinuous correspondence with respect to D and $\overline{T^V}$ is convex nonempty valued for each open absolutely convex symmetric neighborhood V of 0 in E. Then, there exists a point $x^* \in D$ such that $x^* \in \overline{T}(x^*)$.*

4. Applications in equilibrium theory

Let I be a nonempty set (the set of agents). For each $i \in I$, let X_i be a nonempty topological vector space representing the set of actions and define $X := \prod_{i \in I} X_i$; let A_i, $B_i : X \to 2^{X_i}$ be the constraint correspondences and P_i be the preference correspondence for the agent i.

Definition 4 (21). *An abstract economy $\Gamma = (X_i, A_i, P_i, B_i)_{i \in I}$ is a family of ordered quadruples (X_i, A_i, P_i, B_i).*

The notion of equilibrium plays a central role in the theory of equilibrium. In the recent years the generalizations of this concept have been made in some directions, several of them enlarging the set of acceptable points. One of these methods is due to Yuan [21], who divided each constraint correspondence into two parts, A and B, because the set of the fixed points of the "small" correspondence may not be rich enough. Another method leads us to the notion of "pseudo-equilibrium" and we will define it further.

Here, for the definition of the equilibrium we follow Yuan [21].

Definition 5 (21). *An equilibrium for Γ is a point $x^* \in X$ such that for each $i \in I$, $x_i^* \in \overline{B}_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$.*

Remark 5. *When, for each $i \in I$, $A_i(x) = B_i(x)$ for all $x \in X$, this abstract economy model coincides with the classical one introduced by Borglin and Keiding in [2]. If, in addition, $\overline{B}_i(x) = cl_{X_i} B_i(x)$ for each $x \in X$, which is the case if B_i has a closed graph in $X \times X_i$, the definition of equilibrium coincides with that one used by Yannelis and Prabhakar in [20].*

Remark 6. *An example of extension of the equilibrium model considering two constraint correspondences A_i,B_i : $X \to 2^{X_i}$ for each player i (with $A_i(x) \subset B_i(x)$, $x \in X$) is the notion of quasi-equilibrium (see [6]) for an abstract economy, which has an analogue in the private ownership economies. Even if Florenzano considers in [6] that the interest of the quasi-equilibrium concept is purely mathematical, she motivates the research of the conditions which guarantee its existence as being very fruitful from a lot of points of view.*

The following example motivates the necessity of Yuan's model of abstract economy with two constraint correspondences and illustrates it by using correspondences for which the assumptions formulated by us hold, but those made by Yannelis and Brahbakar [20] or by other authors (and which concern the lower semicontinuity) do not hold.

Example 5. *Let $\Gamma = (X, A, B, P)$ be an abstract economy with one agent, where $X = [0, 4]$ and $A, B, P : X \to 2^X$ are defined below:*

$$A(x) = \begin{cases} [1-x, 2] \text{ if } x \in [0, 1); \\ [3, 4] \quad \text{if} \quad x = 1; \\ [0, \frac{1}{2}] \text{ if } \quad x \in (1, 4]; \end{cases}$$
$$P(x) = \begin{cases} [\frac{3}{2}, 2] \quad \text{if } x \in [0, 1); \\ \{4\} \quad \text{if} \quad x = 1; \\ [1, 2] \quad \text{if} \quad x \in (1, 4]; \end{cases}$$
$$B(x) = \begin{cases} [1-x, 2] \text{ if } x \in [0, 1); \\ [3, 4] \quad \text{if} \quad x = 1; \\ [0, 2] \quad \text{if} \quad x \in (1, 4]. \end{cases}$$

The fixed point set of A is $\text{Fix}(A) = [\frac{1}{2}, 1)$, the fixed point set of B is $\text{Fix}(B) = [\frac{1}{2}, 1) \cup (1, 2]$ and $A(x) \subseteq B(x)$ for each $x \in [0, 4]$. Since $U = \{x \in X : (A \cap P)(x) = \emptyset) = [1, 4]$ and $\text{Fix}(A) \cap U = \emptyset$, Yannelis-Prahbakar's model ([20]) (X, A, P) has not equilibrium points.

We notice that $x^* = \frac{3}{2}$ is an equilibrium point for Yuan's model (X, A, B, P) : $(A \cap P)(\frac{3}{2}) = \emptyset$, $\frac{3}{2} \in B(\frac{3}{2})$. The correspondence A proves to not have enough fixed points and it must be enlarged by the correspondence B.

B, P and A are almost w-lower semicontinuous with respect to $D = [0, 2]$ and the game (X, A, B, P) has equilibrium points, as we showed above.

Since $P^{-1}(4) = \{x \in X : 4 \in P(x)\} = \{1\}$, the correspondence P has not open lower sections. We also see that the correspondences A, B and P are not lower semicontinuous on $[0, 4]$.

There is large literature concerning the existence of the equilibrium in Yuan's sense, which has been developed in the last decades. The authors tried to generalize the properties of the involved correspondences; for an overview, see for example [15] or [21]. In [21], Yuan provides applications of abstract economies with two constraint correspondences to the systems of generalized quasi-variational inequalities and to the systems of Ky Fan minimax inequalities.

We define the following type of equilibrium for an abstract economy, which is a slight extension of Yuan's equilibrium. The motivation of introducing it is a mathematical one.

Definition 6. *A pseudo equilibrium for* Γ *is defined as a point* $x^* \in X$ *such that for each* $i \in I$, $x_i^* \in \overline{B}_i(x^*)$ *and* $x^* \in cl\{x \in X : (A_i \cap P_i)(x) = \emptyset\}$ *for each* $i \in I$.

Example 6. *Let* $\Gamma = (X, A, B, P)$ *be an abstract economy with one agent, where* $X = [0, 4]$ *and* $A, B, P : X \to 2^X$ *are defined below:*

$$A(x) = \begin{cases} [1-x, 2] \text{ if } x \in [0,1); \\ [1,4] \quad \text{if} \quad x = 1; \\ [0, \frac{1}{2}] \text{ if } \quad x \in (1,4]; \end{cases}$$

$$P(x) = \begin{cases} [\frac{3}{2}, 2+x] \text{ if } x \in [0,1); \\ \{1\} \quad \text{if} \quad x = 1; \\ [1,2] \quad \text{if} \quad x \in (1,4]; \end{cases}$$

$$B(x) = \begin{cases} [1-x, 2] \text{ if } x \in [0,1); \\ [1,4] \quad \text{if} \quad x = 1; \\ [0,2] \quad \text{if} \quad x \in (1,4]. \end{cases}$$

$$(A \cap P)(x) = \begin{cases} [\frac{3}{2}, 2] \quad \text{if} \quad x \in [0,1); \\ \{1\} \quad \text{if} \quad x = 1; \\ \emptyset \quad \text{if} \quad x \in (1,4]. \end{cases}$$

We note that $x^* = 1$ and $x^* = \frac{3}{2}$ are pseudo-equilibrium points for Γ, since $1 \in B(1)$, $\frac{3}{2} \in B(\frac{3}{2})$ and $1, \frac{3}{2} \in$cl$\{x \in X : (A \cap P)(x) = \emptyset\} = [1, 4]$.

Having a utility function $u_i : X \times X_i \to \mathbb{R}$ for each agent i, we can define a preference correspondence P_i: $P_i(x) := \{y_i \in X_i : u_i(x, y_i) > u_i(x, x_i)\}$. Then, the condition of maximizing the utility function to obtain the equilibrium point becomes: $A_i(x) \cap P_i(x) = \emptyset$ for each $i \in I$.

Now we give an example of correspondence P which is w-lower semicontinuous with respect to a given set, P being constructed from a utility function.

Example 7. *Let G be the game $([-1,1], A, P)$ with $I = \{1\}$, $A : [-1,1] \to 2^{[-1,1]}$ be defined as*

$$A(x) = \begin{cases} [0,1], & \text{if } x \in [-1,0); \\ (\frac{1}{2}, 1] & x \in [0,1]. \end{cases} \text{ and}$$

let $u : [-1,1] \times [-1,1] \to \mathbb{R}$ be the function with levels defined as

$$u(x,y) = \begin{cases} 1, & \text{if } (x,y) \in [-1,0) \times [-1,1) \cup [0,1] \times [-1,0] \backslash \{(0,0)\}; \\ 2 & \text{if } (x,y) \in [-1,0) \times \{(1)\}; \\ 3 & \text{if } (x,y) \in \{(0,0)\} \cup \{0\} \times [0, \frac{1}{2}); \\ 4 & \text{if } (x,y) \in \{0\} \times [\frac{1}{2}, 1]; \\ 5 & \text{if } (x,y) \in (0,1] \times (0,1); \\ 6 & \text{if } (x,y) \in (0,1] \times \{1\}. \end{cases}$$

Then $P : [-1,1] \to 2^{[-1,1]}$ is defined as

$P(x) := \{y \in X : u(x,y) > u(x,x)\}$

$$P(x) = \begin{cases} \{1\}, & \text{if } x \in [-1,0) \cup (0,1); \\ [\frac{1}{2}, 1] & \text{if } x = 0; \\ \phi & \text{if } x = 1. \end{cases}$$

P is not lower semicontinuous.

Let $V = (-\varepsilon, \varepsilon)$ and $D = [1,2]$.

$$P(x) + V = \begin{cases} (1-\varepsilon, 1+\varepsilon), & \text{if } x \in [-1,0) \cup (0,1); \\ (\frac{1}{2} - \varepsilon, 1+\varepsilon) & \text{if } x = 0; \\ \phi & \text{if } x = 1. \end{cases}$$

For $\varepsilon \in (0,1]$,

$$P^V(x) = \begin{cases} [1, 1+\varepsilon), & \text{if } x \in [-1,1); \\ \phi & \text{if } x = 1. \end{cases}$$

For $\varepsilon > 1$,

$$P^V(x) = \begin{cases} [1,2], & \text{if } x \in [-1,1); \\ \phi & \text{if } x = 1. \end{cases}$$

P is w-lower semicontinuous with respect with $D = [1,2]$.

$$A(x) \cap P(x) = \begin{cases} \{1\}, & \text{if } x \in [-1,0) \cup (0,1); \\ (\frac{1}{2}, 1] & \text{if } x = 0; \\ \phi & \text{if } x = 1. \end{cases}$$

$A(1) \cap P(1) = \phi$, and, since $1 \in A(1)$, we have that $x^* = 1$ is an equilibrium point for G.

As an application of the fixed point theorem proved in Section 3, we have the following result.

Theorem 9. *Let $\Gamma = \{X_i, A_i, B_i, P_i\}_{i\in I}$ be an abstract economy such that for each $i \in I$, the following conditions are fulfilled:*

1) *X_i is a nonempty convex subset of a Hausdorff locally convex topological vector space E_i and D_i is a nonempty compact convex metrizable subset of X_i;*

2) *for each $x \in X = \prod_{i\in I} X_i$, $A_i(x)$ and $P_i(x)$ are convex and $B_i(x)$ is nonempty, convex and $A_i(x) \subset B_i(x)$;*

3) *$W_i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is closed in X;*

4) *$H_i : X \to 2^{X_i}$ defined by $H_i(x) = A_i(x) \cap P_i(x)$ for each $x \in X$ is almost w-lower semicontinuous with respect to D_i on W_i* and *$\overline{H_i^{V_i}}$ is convex nonempty valued for each open absolutely convex symmetric neighbourhood V_i of 0 in E_i;*

5) *$B_i : X \to 2^{X_i}$ is almost w-lower semicontinuous with respect to D_i* and *$\overline{B_i^{V_i}}$ is convex nonempty valued for each open absolutely convex symmetric neighbourhood V_i of 0 in E_i;*

6) *for each $x \in X$, $x_i \notin \overline{(A_i \cap P_i)}(x)$.*

*Then there exists $x^* \in D = \prod_{i\in I} D_i$ such that $x_i^* \in \overline{B}_i(x^*)$ and $x^* \in$*cl$\{x \in X : (A_i \cap P_i)(x) = \emptyset\}$ for each $i \in I$ (x^* is a pseudo equilibrium point for Γ).

Proof. Let $i \in I$. By condition 3) we know that W_i is closed in X.

Let's define $T_i : X \to 2^{X_i}$ by $T_i(x) = \begin{cases} A_i(x) \cap P_i(x), & \text{if } x \in W_i, \\ B_i(x), & \text{if } x \notin W_i \end{cases}$ for each $x \in X$.

Then $T_i : X \to 2^{X_i}$ is a correspondence with nonempty convex values. We shall prove that $T_i : X \to 2^{D_i}$ is almost w-lower semicontinuous with respect to D_i. Let ß$_i$ be a basis of open absolutely convex symmetric neighbourhoods of 0 in E_i and let ß$=\prod_{i\in I}$ß$_i$. For each $V = (V_i)_{i\in I} \in \prod_{i\in I}$ß$_i$, for each $x \in X$, let for each $i \in I$

$B^{V_i}(x) = (B_i(x) + V_i) \cap D_i$,

$F^{V_i}(x) = ((A_i(x) \cap P_i(x)) + V_i) \cap D_i$ and

$T_i^{V_i}(x) = \begin{cases} F^{V_i}(x), & \text{if } x \in W_i, \\ B^{V_i}(x), & \text{if } x \notin W_i. \end{cases}$

For each closed set V_i' in D_i, the set

$\left\{x \in X : \overline{T_i^{V_i}}(x) \subset V_i'\right\} =$

$= \left\{x \in W_i : \overline{F^{V_i}}(x) \subset V_i'\right\} \cup \left\{x \in X \setminus W_i : \overline{B^{V_i}}(x) \subset V_i'\right\}$

$= \left\{x \in W_i : \overline{F^{V_i}}(x) \subset V_i'\right\} \cup \left\{x \in X : \overline{B^{V_i}}(x) \subset V_i'\right\}.$

According to condition 6), the set $\left\{x \in W_i : \overline{F^{V_i}}(x) \subset V_i'\right\}$ is closed in X. The set $\left\{x \in X : \overline{B^{V_i}}(x) \subset V_i'\right\}$ is closed in X because $\overline{B^{V_i}}$ is lower semicontinuous.

Therefore, the set $\left\{x \in X : \overline{T_i^{V_i}}(x) \subset V_i'\right\}$ is closed in X. It shows that $\overline{T_i^{V_i}} : X \to 2^{D_i}$ is lower semicontinuous. According to Theorem 2, there exists $x^* \in D = \prod_{i\in I} D_i$ such that $x^* \in \overline{T}_i(x^*)$, for each $i \in I$. By condition 6) we have that $x_i^* \in \overline{B}_i(x^*)$ and $x^* \in$cl$\{x \in X : (A_i \cap P_i)(x) = \emptyset\}$ for each $i \in I$.

Example 8. *Let $\Gamma = \{X_i, A_i, B_i, P_i\}_{i\in I}$ be an abstract economy, where $I = \{1, 2, ..., n\}$, $X_i = [0; 4]$ be a compact convex choice set, $D_i = [0, 2]$ for each $i \in I$ and $X = \prod_{i\in I} X_i$.*

Let $A = \{x \in X : \forall j \in \{1, 2, ..., n\},\ x_j \in [0, 1]$ and $\exists j \in \{1, 2, ..., n\}$ such that $x_j = 1\}$ and let the correspondences $A_i, B_i, P_i : X \to 2^{X_i}$ be defined as follows:

for each $(x_1, x_2, ..., x_n) \in X$,

$$A_i(x) = \begin{cases} [1 - x_i, 2] \text{ if } x \in X \text{ and } \forall j \in \{1, 2, ..., n\},\ x_j \in [0, 1); \\ [3, 4] \quad \text{if} \quad x \in A; \\ [0, \frac{1}{2}], \quad \text{otherwise}; \end{cases}$$

$$P_i(x) = \begin{cases} [\frac{3}{2}, 2 + x_i] \text{ if } x \in X \text{ and } \forall j \in \{1, 2, ..., n\},\ x_j \in [0, 1); \\ \{4\} \quad \text{if} \quad x \in A; \\ [1, 2], \quad \text{otherwise}; \end{cases}$$

$$B_i(x) = \begin{cases} [1 - x_i, 2] \text{ if } x \in X \text{ and } \forall j \in \{1, 2, ..., n\},\ x_j \in [0, 1); \\ [3, 4] \quad \text{if} \quad x \in A; \\ [0, 2], \quad \text{otherwise}. \end{cases}$$

The correspondences A_i, B_i, P_i are not lower semicontinuous on X.

$$A_i(x) \cap P_i(x) = \begin{cases} [\frac{3}{2}, 2] \text{ if } x \in X \text{ and } \forall j \in \{1, 2, ..., n\},\ x_j \in [0, 1); \\ \{4\} \quad \text{if} \quad x \in A; \\ \phi, \quad \text{otherwise}. \end{cases}$$

$W_i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\} = [0, 1]^n$ is closed in X.

$$\overline{(A_i \cap P_i)}(x) = \begin{cases} [\frac{3}{2}, 2] \text{ if } x \in X \text{ and } \forall j \in \{1, 2, ..., n\},\ x_j \in [0, 1); \\ [\frac{3}{2}, 2] \cup \{4\} \quad \text{if} \quad x \in A; \\ \phi, \quad \text{otherwise}. \end{cases}$$

We notice that for each $x \in X$, $x_i \notin \overline{(A_i \cap P_i)}(x)$.

We shall prove that B_i and $(A_i \cap P_i)_{W_i}$ are almost w-lower semicontinuous with respect to $D_i = [0, 2]$.

On W_i,

$$(A_i \cap P_i)(x) = \begin{cases} [\frac{3}{2}, 2] \text{ if } x \in X,\ \forall j \in \{1, 2, ..., n\},\ x_j \in [0, 1); \\ \{4\} \qquad\qquad \text{if} \qquad\qquad x \in A; \end{cases}$$

$$(A_i \cap P_i)(x) + (-\varepsilon, \varepsilon) = \begin{cases} (\frac{3}{2} - \varepsilon, 2 + \varepsilon) \text{ if } x \in X,\ \forall j \in \{1, 2, ..., n\},\ x_j \in [0, 1); \\ (4 - \varepsilon, 4 + \varepsilon) \qquad\qquad \text{if} \qquad\qquad x \in A; \end{cases}$$

Let $(A_i \cap P_i)^V(x) = ((A_i \cap P_i)(x) + (-\varepsilon, \varepsilon)) \cap [0, 2]$, where $V = (-\varepsilon, \varepsilon)$.

Then,

if $\varepsilon \in (0, \frac{3}{2}]$,

$$(A_i \cap P_i)^V(x) = \begin{cases} (\frac{3}{2} - \varepsilon, 2] \text{ if } x \in X \text{ and } \forall j \in \{1, 2, ..., n\},\ x_j \in [0, 1); \\ \phi \qquad\qquad \text{if} \qquad\qquad x \in A; \end{cases}$$

if $\varepsilon \in (\frac{3}{2}, 2]$,

$$(A_i \cap P_i)^V(x) = \begin{cases} [0, 2] \text{ if } x \in X \text{ and } \forall j \in \{1, 2, ..., n\},\ x_j \in [0, 1); \\ \phi \qquad\qquad \text{if} \qquad\qquad x \in A; \end{cases}$$

if $\varepsilon \in (2, 4]$,

$$(A_i \cap P_i)^V(x) = \begin{cases} [0, 2] \text{ if } x \in X \text{ and } \forall j \in \{1, 2, ..., n\},\ x_j \in [0, 1); \\ (4 - \varepsilon, 2] \qquad\qquad \text{if} \qquad\qquad x \in A; \end{cases}$$

and if $\varepsilon > 4$,

$(A_i \cap P_i)^V(x) = [0, 2]$ if $x \in [0, 1]^n$.

Hence, for each $V = (-\varepsilon, \varepsilon)$, $\overline{(A_i \cap P_i)^V}_{W_i}$ is lower semicontinuous and has nonempty values.

$$B_i(x) + (-\varepsilon, \varepsilon) = \begin{cases} (1 - x_i - \varepsilon,\ 2 + \varepsilon) \text{ if } x \in X,\ \forall j \in \{1, 2, ..., n\},\ x_j \in [0, 1); \\ (3 - \varepsilon,\ 4 + \varepsilon) \qquad\qquad \text{if} \qquad\qquad x \in A; \\ (-\varepsilon,\ 2 + \varepsilon) \qquad\qquad \text{otherwise.} \end{cases}$$

Let $B_i^V(x) = (B_i(x) + (-\varepsilon, \varepsilon)) \cap [0, 2]$, where $V = (-\varepsilon, \varepsilon)$.

Then,

if $\varepsilon \in (0, 1]$,

$$B_i^V(x) = \begin{cases} (1 - x_i - \varepsilon, 2] & \text{if } x \in X,\ x_i \in [0, 1 - \varepsilon], \\ & \forall j \in \{1, 2, ..., n\} \setminus \{i\},\ x_j \in [0, 1); \\ [0, 2] & \text{if } x \in X,\ x_i \in (1 - \varepsilon, 1), \\ & \forall j \in \{1, 2, ..., n\} \setminus \{i\},\ x_j \in [0, 1); \\ \phi & \text{if} \quad x \in A; \\ [0, 2] & \text{otherwise;} \end{cases}$$

$$\text{if } \varepsilon \in (1, 3],\ B_i^V(x) = \begin{cases} [0, 2] \text{ if } x \in X \text{ and } \forall j \in \{1, 2, ..., n\},\ x_j \in [0, 1); \\ (3 - \varepsilon, 2] \qquad\qquad \text{if} \qquad\qquad x \in A; \\ [0, 2] \qquad\qquad \text{otherwise.} \end{cases}$$

and if $\varepsilon > 3$, ${B_i}^V(x) = [0, 2]$ if $x \in X$.

Then, for each $V = (-\varepsilon, \varepsilon)$, $\overline{B_i^V}$ is lower semicontinuous and has nonempty values.

Therefore, all hypotheses of Theorem 3 are satisfied, so that there exists an equilibrium point $x^* = \{\frac{3}{2}, \frac{3}{2}, ..., \frac{3}{2}\} \in X$ such that $x_i^* \in \overline{B}_i(x^*)$ and $(A_i \cap P_i)(x^*) = \emptyset$.

Theorem 4 deals with abstract economies which have dual w-lower semicontinuous pairs of correspondences and can be compared with Theorem 5 in Wu [19].

Theorem 10. *Let $\Gamma = \{X_i, A_i, B_i, P_i\}_{i \in I}$ be an abstract economy such that for each $i \in I$, the following conditions are fulfilled:*

1) *X_i is a nonempty convex subset of a Hausdorff locally convex topological vector space E_i* and *D_i is a nonempty compact convex metrizable subset of X_i;*

2) *for each $x \in X = \prod_{i \in I} X_i$, $P_i(x) \subset D_i$, and $B_i(x)$ is nonempty;*

3) *the set $W_i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X;*

4) *the pair $(A_{i|\mathrm{cl}W_i}, P_{i|\mathrm{cl}W_i})$ is dual almost w-lower semicontinuous with respect to D_i, $B_i : X \to 2^{X_i}$ is almost w-lower semicontinuous with respect to D_i;*

5) *if $T_{i,V_i} : X \to 2^{X_i}$ is defined by $T_{i,V_i}(x) = (A_i(x) + V_i) \cap D_i \cap P_i(x)$ for each $x \in X$, then the correspondences $\overline{B_i^{V_i}}$ and $\overline{T_{i,V_i}}$ are nonempty convex valued for each open absolutely convex symmetric neighbourhood V_i of 0 in E_i;*

6) *for each $x \in X$, $x_i \notin \overline{P}_i(x)$;*

Then, there exists $x^* \in D = \prod_{i \in I} D_i$ such that $x_i^* \in \overline{B}_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$ for all $i \in I$.

Proof. For each $i \in I$, let ß$_i$ denote the family of all open absolutely convex symmetric neighbourhoods of zero in E_i and let ß$= \prod_{i \in I}$ß$_i$. For each $V = \prod_{i \in I} V_i \in \prod_{i \in I}$ß$_i$, for each $i \in I$, let

$B^{V_i}(x) = (B_i(x) + V_i) \cap D_i$ for each $x \in X$ and

$$S_i^{V_i}(x) = \begin{cases} T_{i,V_i}(x), & \text{if } x \in \mathrm{cl}W_i, \\ B_i^{V_i}(x), & \text{if } x \notin \mathrm{cl}W_i, \end{cases}$$

$\overline{S_i^{V_i}}$ has closed values. Next, we shall prove that $\overline{S_i^{V_i}} : X \to 2^{D_i}$ is lower semicontinuous.

For each closed set V' in D_i, the set

$\left\{x \in X : \overline{S_i^{V_i}}(x) \subset V'\right\} =$

$= \left\{x \in \mathrm{cl}W_i : \overline{T_{i,V_i}}(x) \subset V'\right\} \cup \left\{x \in X \setminus \mathrm{cl}W_i : \overline{B_i^{V_i}}(x) \subset V'\right\}$

$=\left\{x \in \mathrm{cl}W_i : \overline{T_{i,V_i}}(x) \subset V'\right\} \cup \left\{x \in X : \overline{B_i^{V_i}}(x) \subset V'\right\}.$

We know that the correspondence $\overline{T_{i,V_i}}(x)_{|\mathrm{cl}W_i} : \mathrm{cl}W_i \to 2^{D_i}$ is lower semicontinuous. The set $\left\{x \in \mathrm{cl}W_i : \overline{T_{i,V_i}}(x) \subset V'\right\}$ is closed in $\mathrm{cl}W_i$, and hence it is also closed in X because $\mathrm{cl}W_i$ is closed in X. Since $\overline{B_i^{V_i}}(x) : X \to 2^{D_i}$ is lower semicontinuous, the set $\{x \in X : \overline{B_i^{V_i}}(x)\} \subset V'$ is closed in X and therefore the set $\left\{x \in X : \overline{S_i^{V_i}}(x) \subset V'\right\}$ is closed in X. It showes that $\overline{S_i^{V_i}} : X \to 2^{D_i}$ is lower semicontinuous. According to Wu's Theorem 1, applied for the correspondences $\overline{S_i^{V_i}} = T_i^{V_i}$, there exists a point $x_V^* \in D = \prod_{i\in I} D_i$ such that $x_{V_i}^* \in T_i^{V_i}(x_V^*)$ for each $i \in I$. By condition 5), we have that $x_{V_i}^* \notin \overline{P_i}(x_V^*)$, hence, $x_{Vi}^* \notin \overline{A_i^{V_i}}(x_V^*) \cap \overline{P_i}(x_V^*)$.

We also have that $\mathrm{clGr}(T_{i,V_i}) \subseteq \mathrm{clGr}(A_i^{V_i}) \cap \mathrm{clGr}P_i$.Then $\overline{T_{i,V_i}}(x) \subseteq \overline{A_i^{V_i}}(x) \cap \overline{P_i}(x)$ for each $x \in X$. It follows that $x_{Vi}^* \notin \overline{T_{i,V_i}}(x_V^*)$. Therefore, $x_{Vi}^* \in \overline{B^{V_i}}(x_V^*)$.

For each $V = (V_i)_{i\in I} \in \prod_{i\in I} ß_i$, let's define $Q_V = \cap_{i\in I}\{x \in D : x \in \overline{B^{V_i}}(x)$ and $A_i(x) \cap P_i(x) = \emptyset\}$.

Q_V is nonempty since $x_V^* \in Q_V$, and it is a closed subset of D according to 3). Then, Q_V is nonempty and compact.

Let ß$=\prod_{i\in I} ß_i$. We prove that the family $\{Q_V : V \in ß\}$ has the finite intersection property.

Let $\{V^{(1)}, V^{(2)}, ..., V^{(n)}\}$ be any finite set of ß and let $V^{(k)} = \prod_{i\in I} V_i^{(k)}{}_{i\in I}$, $k = 1, ..., n$. For each $i \in I$, let $V_i = \bigcap_{k=1}^{n} V_i^{(k)}$, then $V_i \in ß_i$; thus $V \in \prod_{i\in I} ß_i$. Clearly $Q_V \subset \bigcap_{k=1}^{n} Q_{V^{(k)}}$ so that $\bigcap_{k=1}^{n} Q_{V^{(k)}} \neq \emptyset$.

Since D is compact and the family $\{Q_V : V \in ß\}$ has the finite intersection property, we have that $\cap\{Q_V : V \in ß\} \neq \emptyset$. Take any $x^* \in \cap\{Q_V : V \in ß\}$, then for each $V \in ß$,

$x^* \in \cap_{i\in I} \left\{ x^* \in D : x_i^* \in \overline{B^{V_i}}(x) \text{ and } A_i(x) \cap P_i(x) = \emptyset) \right\}.$

Hence, $x_i^* \in \overline{B^{V_i}}(x^*)$ for each $V \in$ß and for each $i \in I$. According to Lemma 1, we have that $x_i^* \in \overline{B}_i(x^*)$ and $(A_i \cap P_i)(x^*) = \emptyset$ for each $i \in I$.

Now we introduce the next concept which also generalizes the lower semicontinuous correspondences.

Let X be a non-empty convex subset of a topological linear space E, Y be a non-empty set in a topological space and $K \subseteq X \times Y$.

Definition 7. *The correspondence $T : X\times Y \to 2^X$ has the e-LSCS-property (e-lower semicontinuous selection property) on K, if for each absolutely convex neighborhood V of zero in E, there exists a lower semicontinuous correspondence with convex values $S^V : X \times Y \to 2^X$ such that $S^V(x,y) \subset T(x,y)+clV$ and $x \notin clS^V(x,y)$ for every $(x,y) \in K$.*

The following theorem concerns the abstract economies which have correspondences with e-LSCS-property.

Theorem 11. *Let $\Gamma = (X_i, A_i, P_i, B_i)_{i\in I}$ be an abstract economy, where I is a (possibly uncountable) set of agents such that for each $i \in I$:*

1) *X_i is a non-empty compact set in a Hausdorff locally convex topological vector space E_i;*
2) *B_i is w-lower semicontinuous with nonempty convex values;*
3) *the set $W_i := \{x \in X \ / \ (A_i \cap P_i)(x) \neq \emptyset\}$ is open;*
4) *$(A_i \cap P_i)$ has the e-LSCS-property on W_i.*

Then there exists an equilibrium point $x^* \in X$ for Γ, i.e., for each $i \in I$, $x_i^* \in \overline{B}_i(x^*)$ and $(A_i \cap P_i)(x^*) = \emptyset$ for each $i \in I$.

Proof. For each $i \in I$, let ß$_i$ denote the family of all open absolutely convex neighborhoods of zero in E_i. Let $V = (V_i)_{i\in I} \in \prod_{i\in I}$ß$_i$. Since $(A_i \cap P_i)$ has the e-LSCS-property on W_i, it follows that there exists a lower semicontinuous correspondence $F_i^{V_i} : X \to 2^{X_i}$ with convex values such that $F_i^{V_i}(x) \subset (A_i \cap P_i)(x) + V_i$ and $x_i \notin \mathrm{cl}F_i^{V_i}(x)$ for each $x \in W_i$.

Let's define the correspondence $T_i^{V_i} : X \to 2^{X_i}$, by

$$T_i^{V_i}(x) := \begin{cases} F_i^{V_i}(x), & \text{if } x \in \mathrm{cl}W_i, \\ (B_i(x) + V_i) \cap X_i, & \text{if } x \notin \mathrm{cl}W_i. \end{cases}$$

$B^{V_i} : X \to 2^{X_i}$, $B^{V_i}(x) = (B_i(x) + V_i) \cap X_i$ is lower semicontinuous according to the assumption 2).

Let U be an closed subset of X_i, then
$U' := \{x \in X \mid T_i^{V_i}(x) \subset U\}$
$=\{x \in \mathrm{cl}W_i \mid T_i^{V_i}(x) \subset U\} \cup \{x \in X \backslash \mathrm{cl}W_i \mid T_i^{V_i}(x) \subset U\}$
$=\left\{x \in \mathrm{cl}W_i \mid F_i^{V_i}(x) \subset U\right\} \cup \{x \in X \mid (B_i(x) + V_i) \cap X_i \subset U\}$

U' is a closed set, because $\mathrm{cl}W_i$ is closed, $\left\{x \in \mathrm{cl}W_i \mid F_i^{V_i}(x) \subset U\right\}$ is closed since $F_i^{V_i}(x)$ is lower semicontinuous map on $\mathrm{cl}W_i$ and the set

$\{x \in X \mid (B_i(x) + V_i) \cap X_i \subset U\}$ is closed since $(B_i(x)+V_i)\cap X_i$ is lower semicontinuous. Then $T_i^{V_i}$ is lower semicontinuous on X. By Theorem 7.3.3 in [8], $\mathrm{cl}T_i^{V_i}$ is lower semicontinuous and has closed convex values.

Since X is a compact convex set, by Wu's fixed-point theorem [18], there exists $x_V^* \in X$ such that for each $i \in I$, $(x_V^*)_i \in \mathrm{cl}T_i^{V_i}(x_V^*)$. If $x_V^* \in \mathrm{cl}W_i$, $(x_V^*)_i \in \mathrm{cl}F_i^{V_i}(x_V^*)$, which is a contradiction.

Hence, $(x_V^*)_i \in \mathrm{cl}[(B_i(x_V^*)+V_i)\cap X_i]$ and $(A_i \cap P_i)(x_V^*) = \emptyset$. We have that $(x_V^*)_i \in \mathrm{cl}[(B_i(x_V^*) + V_i) \cap X_i] \subset \overline{B^{V_i}}(x_V^*)$.

Let's define $Q_V = \cap_{i\in I}\{x \in X : x_i \in \overline{B^{V_i}}(x) \text{ and } (A_i \cap P_i)(x) = \emptyset\}$. We have that $x_V^* \in Q_V$, then $Q_V \neq \emptyset$ and so it is a non-empty closed subset of X by 3) and hence Q_V is compact.

We prove that the family $\{Q_V : V \in \prod_{i\in I} ß_i\}$ has the finite intersection property.

Let $\{V^{(1)}, V^{(2)}, ..., V^{(n)}\}$ be any finite set of $\prod_{i\in I} ß_i$ and let $V^{(k)} = \prod_{i\in I} V_i^{(k)}$, where $V_i^{(k)} \in ß_i$ for each $i \in I$. Let $V_i = \bigcap_{k=1}^{n} V_i^{(k)}$, then $V_i \in ß_i$; thus $V = \prod_{i\in I} V_i \in \prod_{i\in I} ß_i$. Clearly, $Q_V \subset \bigcap_{k=1}^{n} Q_{V^{(k)}}$ so that $\bigcap_{k=1}^{n} Q_{V^{(k)}} \neq \emptyset$.

Therefore, the family $\{Q_V : V \in \prod_{i\in I} ß_i\}$ has the finite intersection property. Since X is compact, we have that $\cap\{Q_V : V \in \prod_{i\in I} ß_i\} \neq \emptyset$. Let's take any $x^* \in \cap\{Q_V : V \in \prod_{i\in I} ß_i\}$, then for each $i \in I$ and each $V_i \in ß_i$, $x_i^* \in \overline{B^{V_i}}(x^*)$ and $(A_i \cap P_i)(x^*) = \emptyset$; but then, $x_i^* \in \overline{B}_i(x^*)$ from Lemma 1 and $(A_i \cap P_i)(x^*) = \emptyset$ for each $i \in I$.

5. The model of a generalized multiobjective game and the existence of generalized Pareto equilibrium

The purpose of this section is to make a preliminary unitary presentation of the model of a constrained multicriteria game in its strategic form and of the solution concepts for this type of game, and also to state an existence result for generalized Pareto equilibria.

Let I be a finite set (the set of players). For each $i \in I$, let X_i be the set of strategies and define $X = \prod_{i \in I} X_i$. Let $T^i : X \to 2^{\mathbb{R}^{k_i}}$, where $k_i \in \mathbb{N}$, be the multicriteria payoff function and let $A^i : X \to 2^{X_i}$ be a constraint correspondence.

Definition 8 (9). *The family $G = (X_i, A^i, T^i)_{i \in I}$ is called a generalized multicriteria (multiobjective) game.*

Any n-tuple of strategies can be regarded as a point in the product space of sets of players' strategies: $x = (x_1, x_2, ...x_n) \in X$. For each player $i \in I$, the vector of the $n-1$ strategies of the other ones will be denoted by $x_{-i} = (x_1, ...x_{i-1}, x_{i+1}, ...x_n) \in X_{-i} = \prod_{j \in I \setminus \{i\}} X_i$. We note that $x = (x_{-i}, x_i)$.

We assume that each player is trying to minimize his/her own payoff according to his/her preferences, where for each player $i \in I$, the preference " $\gtrsim_i$ " over the outcome space $\mathbb{R}^{k_i}$ is the following:

$z^1 \gtrsim_i z^2$ if only if $z^1_j \geq z^2_j$ for each $j = 1, 2, ...k_i$ and $z^1, z^2 \in \mathbb{R}^{k_i}$. The following preference can be defined on X for each player i (see [9]):

$x \gtrsim_i y$ whenever $F^i(x) \gtrsim_i F^i(y)$ and $x, y \in X$.

Let $x^* = (x^*_1, x^*_2, ..., x^*_n) \in X$.

We introduce slight generalizations of the equilibrium concepts defined by Kim and Ding in [9].

Definition 9. *A strategy $x^*_i \in X_i$ of player i is said to be a generalized Pareto efficient strategy (respectively, a weak Pareto efficient strategy) with respect to x if $x^*_i \in \overline{A^i}(x^*)$ and there is no strategy $x_i \in A^i(x^*)$ such that*

$$T^i(x^*) - T^i(x^*_{i-1}, x_i) \in \mathbb{R}^{k_i}_+ \setminus \{0\} \text{ (respectively, } T^i(x^*) - T^i(x^*_{i-1}, x_i) \in \mathrm{int}\mathbb{R}^{k_i}_+ \setminus \{0\}).$$

Definition 10. *A strategy $x^* \in X$ is said to be a generalized Pareto equilibrium (respectively, a weak Pareto equilibrium) of a game $G = (X_i, A^i, T^i)_{i \in I}$, if for each player $i \in I$, $x^*_i \in X_i$ is a Pareto efficient strategy against x^* (respective, a generalized weak Pareto efficient strategy against x^*).*

The following notion contains the idea of a game equilibrium defined by using a scalarization function. In this case, the scalarization method uses weighted coefficients W_i, so that each player i has his own vector of weights $W_i \in \mathbb{R}_+^{k_i} \backslash \{0\}$.

Definition 11. *A strategy $x^* \in X$ is said to be a generalized weighted Nash equilibrium with respect to the weighted vector $W = (W_i)_{i \in I}$ with $W_i =$ $(W_{i,1}, W_{i,2},$*

$..., W_{i,k_i}) \in \mathbb{R}_+^{k_i}$ of the multiobjective game $G = ((X_i, A^i, T^i)_{i \in I}$ if for each player $i \in I$, we have

1) $x_i^* \in \overline{A^i}(x^*)$;

2) $W_i \in \mathbb{R}_+^{k_i} \backslash \{0\}$;

3) for all $x_i \in A^i(x^*)$, $W_i \cdot T^i(x^*) \leq W_i \cdot T^i(x_{-i}^*, x_i)$, where $\cdot$ denotes the inner product in $\mathbb{R}^{k_i}$.

Remark 7. *In particular, if $W_i \in \Delta^{k_i} = \{u_i \in \mathbb{R}_+^{k_i}$ with $\sum_{j=1}^{k_i} u_{i,j} = 1\}$ for each $i \in I$, then the strategy $x^* \in X$ is said to be a normalized generalized weighted Nash equilibrium with respect to W.*

Remark 8. *If for each $i \in I$, $\overline{A^i}$ has closed values and a closed graph in $X \times X_i$, the notions of equilibrium introduced above coincide with the equilibrium notions defined by Kim and Ding in [9].*

The relationship between the two types of equilibrium notions is given by the following result.

Lemma 12. *Each normalized generalized weighted Nash equilibrium $x^* \in X$ with a weight $W = (W_1, ...W_n) \in \Delta^{k_1} \times ... \times \Delta^{k_n}$ (respectively, $W = (W_1, ..., W_n) \in int\Delta^{k_1} \times ... \times int\Delta^{k_n}$) is a weak Pareto equilibrium (respectively, a Pareto equilibrium) of the game $G = ((X_i, A^i, T^i)_{i \in I}$.*

The proof follows the same line as the proof of Lemma 7 in [9].

Remark 9. *As in [9], the above lemma remains true when $W = (W_1, ..., W_n)$ satisfies $W_i \in \mathbb{R}_+^{k_i}$ (resp. $W_i \in int\mathbb{R}_+^{k_i}$).*

In order to prove the existence result for generalized weighted Nash equilibrium of generalized multiobjective games, first we prove the following lemma.

Lemma 13. *Let X be a nonempty convex compact of a Hausdorff locally convex topological vector space E, D be a nonempty compact convex and metrizable subset of X, $A : X \to 2^X$ be a correspondence with non-empty convex values and $f : X \times X \to \overline{\mathbb{R}}$ be a function such that:*

1) *A is almost weakly lower semicontinuous with respect to D and $\overline{A^V}$ is convex nonempty valued for each open absolutely convex symmetric neighbourhood V of 0 in E;*

2) *The correspondence $F : X \to 2^X$, $F(x) = \{y \in X : f(x,x) - f(y,x) > 0\}$ is almost weakly lower semicontinuous with respect to D on $K = \{x \in X : x \in \overline{A}(x)\}$ and $\overline{F^V}$ is convex valued for each open absolutely convex symmetric neighbourhood V of 0 in E;*

3) *$x \notin \overline{F}(x)$ for each $x \in K$;*

then there exists $x^* \in X$ such that $x^* \in \overline{A}(x^*)$ and $f(x^*, x^*) \leq f(y, x^*)$ for each $y \in \overline{A}(x^*)$.

Proof. We notice first that the set $K = \{x \in X : x \in \overline{A}(x)\}$ is closed.

Assume that for each $x \in K$, $A(x) \cap F(x) \neq \phi$ and define the correspondence $G : X \to 2^X$ by

$$G(x) = \begin{cases} A(x) \cap F(x) \text{ if } x \in K; \\ A(x) \quad \text{ if } \quad x \notin K. \end{cases}$$

By 1) and 3), the correspondence $\overline{G^V} : X \to 2^X$ is lower semicontinuous for each open absolutely convex symmetric neighbourhood V of 0 in E, and has nonempty convex closed values (we can prove this fact by using an argument similar with that one from the Theorem 3). By Corollary 2, there exists $x^* \in X$ such that $x^* \in \overline{G}(x^*)$. By definition of G and A, x^* must be in K. It follows that $x^* \in \overline{A \cap F}(x^*)$, and since clGr$A \cap F \subset$clGr$A \cap$clGr$F$, we have that $x^* \in \overline{A}(x^*) \cap \overline{F}(x^*)$, that is $x^* \in \overline{F}(x^*)$, which contradicts 3). Therefore, there exists $x^* \in K$ such that $\overline{A}(x^*) \cap \overline{F}(x^*) = \phi$ (this implies also $A(x^*) \cap F(x^*) = \phi$). Hence

$$x^* \in \overline{A}(x^*) \text{ and } f(x^*, x^*) \leq f(y, x^*) \text{ for each } y \in \overline{A}(x^*).$$

Example 9. *Let $f : [-1,1] \times [-1,1] \to \mathbb{R}$,*

$$f(x,y)=\begin{cases} 1 & \text{if} \quad (x,y)=(-1,0); \\ 2 & \text{if} \quad (x,y)=(0,0); \\ 1 & \text{if} \quad x,y\in[0,1)\times[0,1]\backslash\{(0,0)\}; \\ 2 & \text{if} \quad (x,y)\in(\frac{1}{2},1]\times[-1,0); \\ 3 & \text{if} \quad (x,y)\in[-1,\frac{1}{2}]\times[-1,0)\cup\{(-1,0)\times\{0\}\}; \\ 4 & \text{if} \quad (x,y)\in[-1,0)\times(0,1]; \\ 0 & \text{if} \quad (x,y)\in\{1\}\times(0,1]; \end{cases}$$

Let $A:[-1,1]\to 2^{[-1,1]}$ defined by $A(x)=[-1,0]$ if $x\in[-1,1]$.

A is lower semicontiuous on $[-1,1]$ and $K=\{x\in[-1,1]: x\in\overline{A}(x)\}=[-1,0]$ is closed.

$F:X\to 2^X$, $F(x)=\{y\in X: f(x,x)-f(y,x)>0\}$

$$F(x)=\begin{cases} (\frac{1}{2},1] \text{ if } x\in[-1,0); \\ \{-1\} \quad \text{if} \quad x=0; \\ \{1\} \quad \text{if} \quad x\in(0,1]. \end{cases}$$

F is not lower semicontinuous and $x\notin\overline{F}(x)$, $\forall x\in K=[-1,0]$, where
$\overline{F}_K(x)=\begin{cases} [\frac{1}{2},1] & \text{if} \quad x\in[-1,0); \\ \{-1\}\cup[\frac{1}{2},1] & \text{if} \quad x=0; \end{cases}$

We also have that for each $V=(-\varepsilon,\varepsilon)$ with $\varepsilon>0$, and $D=[0,1]$, $F^V_{|K}$ is lower semicontinuous.

Therefore,

$F_{|K}(x)+(-\varepsilon,\varepsilon)=\begin{cases} (\frac{1}{2}-\varepsilon,1+\varepsilon) \text{ if } x\in[-1,0); \\ (-1-\varepsilon,-1+\varepsilon) \quad \text{if} \quad x=0; \end{cases}$

For $\varepsilon\in(0,\frac{1}{2}]$, $F^V_{|K}(x)=\begin{cases} (\frac{1}{2}-\varepsilon,1] \text{ if } x\in[-1,0); \\ \phi \quad \text{if} \quad x=0; \end{cases}$

For $\varepsilon\in(\frac{1}{2},1]$, $F^V_{|K}(x)=\begin{cases} [0,1] \quad \text{if } x\in[-1,0); \\ \phi \quad \text{if} \quad x=0; \end{cases}$

For $\varepsilon\in(1,2]$, $F^V_{|K}(x)=\begin{cases} [0,1] \text{ if } x\in[-1,0); \\ [0,-1+\varepsilon) \text{ if } x=0; \end{cases}$

For $\varepsilon>2$, $F^V_{|K}(x)=[0,1]$, $x\in[-1,0]$.

Then,

For $\varepsilon\in(0,\frac{1}{2}]$, $\overline{F^V_{|K}}(x)=[\frac{1}{2}-\varepsilon,1]$ if $x\in[-1,0]$;

For $\varepsilon>\frac{1}{2}$ $\overline{F^V_{|K}}(x)=[0,1]$ if $x\in[-1,0]$;

$\overline{F^V_{|K}}$ is lower semicontinuous and nonempty convex valued.

By Lemma 3, we have that there is $x^*\in\overline{A}(x^*)$ such that $A(x^*)\cap F(x^*)=\phi$.

For example, $x^*=-\frac{1}{2}$, -$\frac{1}{2}\in\overline{A}(-\frac{1}{2})$ and -$\frac{1}{2}\notin F(-\frac{1}{2})$, that is $3=$

$f(-\frac{1}{2},-\frac{1}{2}) \geq f(y,-\frac{1}{2}) = 3$ for each $y \in \overline{A}(-\frac{1}{2}) = [-1,0]$.

Now, as an application of Lemma 3, we have the following existence theorem of generalized weighted Nash equilibrium for generalized multiobjective games.

Theorem 14. *Let I be a finite set of indices, let $(X_i, A^i, T^i)_{i\in I}$ be a constrained multi-criteria game with for each $i \in I$, X_i is a nonempty convex subset of a Hausdorff locally convex topological vector space E^i and suppose that there is a nonempty compact convex and metrizable subset D of $X = \prod_{i\in I} X_i$ and a weighted vector $W = (W_1, W_2, ..., W_n)$ with $W_i \in \mathbb{R}^{k_i}_+ \backslash \{0\}$ such that the following conditions are satisfied:*

1) *for each $i \in I$, A^i is almost weakly lower semicontinuous with respect to D and $\overline{A^{i,V_i}}$ is convex nonempty valued for each open absolutely convex symmetric neighbourhood V_i of 0 in E_i;*

2) *The set $K = \{x \in X : x \in \overline{A}(x)\}$, where $A(x) = \prod_{i\in I} A^i(x)$, is closed in X;*

3) *The correspondence $F : X \to 2^X$, $F(x) = \{y \in X : \sum_{i=1}^n W_i \cdot (T^i(x_{-i}, x_i) - T^i(x_{-i}, y_i)) > 0\}$ is almost weakly lower semicontinuous with respect to D on K and $\overline{F^V}$ is convex valued for each open absolutely convex symmetric neighbourhood V of 0 in E;*

4) *$x \notin \overline{F}(x)$ for each $x \in K$;*

then there exists $x^* \in X$ such that x^* is a generalized weighted Nash equilibria with respect to W.

Proof. Define the function $f : X \times X \to \mathbb{R}$ by $f(x,y) = \sum_{i=1}^n W_i \cdot (T^i(x_{-i}, x_i) - T^i(x_{-i}, y_i))$, $(x,y) \in X \times X$. It is easy to see that f satisfies all hypothesis of Lemma 3, hence there exists $x^* \in X$ such that $x^* \in \overline{A}(x^*)$ and $\sum_{i=1}^n W_i \cdot (T^i(x^*_{-i}, x^*_i) - T^i(x^*_{-i}, y_i) \leq 0$ for any $y \in \overline{A}(x^*)$. We use the fact that $\prod_{i\in I} A^i \subseteq \overline{\prod_{i\in I} A^i} \subseteq \prod_{i\in I} \overline{A^i}$. We obtain first $x^*_i \in \overline{A}_i(x^*)$ for each $i \in I$. For any given $i \in I$ and any given $y_i \in A^i(x^*)$, let $y = (x^*_{-i}, y_i)$. Then

$W_i \cdot (T^i(x^*_{-i}, x^*_i) - T^i(x^*_{-i}, y_i)) =$

$= \sum_{j=1}^n W_j \cdot (T^j(x^*_{-i}, x^*_i) - T^i(x^*_{-i}, y_i)) - \sum_{j\neq i} W_j \cdot (T^j(x^*_{-i}, x^*_i) - T^i(x^*_{-i}, y_i))$

$= \sum_{j=1}^n W_j \cdot (T^j(x^*_{-i}, x^*_i) - T^i(x^*_{-i}, y_i)) \leq 0.$

Therefore, we have $W_i \cdot (T^i(x^*_{-i}, x^*_i) - T^i(x^*_{-i}, y_i)) \leq 0$ for each $i \in I$ and $y_i \in A^i(x^*)$. Hence, x^* is a generalized weighted Nash equilibrium of the game G with respect to W.

By using Lemma 3, we obtain the following existence theorem of generalized Pareto equilibrium as a consequence of Theorem 6.

Theorem 15. *Let I be a finite set of indices, let $(X_i, A^i, T^i)_{i\in I}$ be a constrained multi-criteria game with for each $i \in I$, X_i is a nonempty convex subset of a Hausdorff locally convex topological vector space E^i and suppose that there is a nonempty compact convex and metrizable subset D of $X = \prod_{i\in I} X_i$ and a weighted vector $W = (W_1, W_2, ..., W_n)$ with $W_i \in \mathbb{R}_+^{k_i} \backslash \{0\}$ such that the following conditions are satisfied:*

1) *for each $i \in I$, A^i is almost weakly lower semicontinuous with respect to D and $\overline{A^{i,V_i}}$ is convex nonempty valued for each open absolutely convex symmetric neighbourhood V_i of 0 in E_i;*
2) *The set $K = \{x \in X : x \in \overline{A}(x)\}$, where $A(x) = \prod_{i\in I} A^i(x)$, is closed in X;*
3) *The correspondence $F : X \to 2^X$, $F(x) = \{y \in X : \sum_{i=1}^n W_i \cdot (T^i(x_{-i}, x_i) - T^i(x_{-i}, y_i)) > 0\}$ is almost weakly lower semicontinuous with respect to D on K and $\overline{F^V}$ is convex valued for each open absolutely convex symmetric V of 0 in E;*
4) *$x \notin \overline{F}(x)$ for each $x \in K$;*

then there exists $x^* \in X$ such that x^* is a generalized weak Pareto equilibrium.

Furthermore, if $W_i \in int R_+^{k_i} \backslash \{0\}$ for all $i \in I$, then x^* is a generalized Pareto equilibrium.

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